Linear Combinants of Systems of Binary Forms, with the Syzygies of the Second Degree Connecting Them.

BY WALTER FRANCIS SHENTON.

The large use that has been made of the theory of combinants in the realm of the invariant theory of rational curves has made the present research seem well worth while. It has been the purpose of this investigation to set forth a comparatively simple and orderly method for obtaining the linear and quadratic combinants of a given system of two or three binary forms of the same order.

After having followed out the work sufficiently far to cover all the cases of rational plane and space curves as far as the sextic, we have arranged these explicit forms in a series of tables for easy reference. By this means we believe that they will be valuable to any person who desires to find the corresponding combinants for any canonical forms.

I. *Combinant Defined.*

A combinant of a system of quantics of the same order is an invariant or covariant of this system which remains unchanged, to within a constant factor, not only when the variables are linearly transformed, but also when for this system of quantics there is substituted another system of quantics which are linear combinations of the original quantics.

In particular, a combinant of a number of binary quantics $f_1, f_2, f_3, \ldots, f_n$, in the same variables, is an invariant or covariant of these quantics which differs only by a power of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & \ldots \\ a_2 & b_2 & c_2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_n & b_n & c_n & \ldots \end{vmatrix}$$

from the same invariant or covariant of $F_1, F_2, F_3, \ldots, F_n$, where

$$F_i = a_i f_1 + b_i f_2 + c_i f_3 + \ldots, \qquad (i=1, 2, 3, \ldots, n).$$

II. *A combinant may be expressed in terms of coefficients which are determinants formed from the matrix of the coefficients of the quantics from which the combinant is derived. Conversely, a covariant or invariant of a system of quantics is a combinant of the system if its coefficients may be expressed in terms of such determinants.*

The first of these theorems is an immediate outgrowth of the fundamental theorem for invariants. A more general statement of this theorem is given

by Gordan.* To develop a proof for the converse, let us consider two binary forms; say, $(at)^n$ and $(bt)^n$. The matrix of their coefficients is

$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \end{vmatrix}.$$

If now we look upon (a_0, b_0), (a_1, b_1), etc., as coordinates of points along a line, the combinants, which may be regarded as associated with the whole line rather than the individual points upon it, will remain invariant under the transformations

$$a_i' = \alpha a_i + \beta b_i, \quad b_i' = \gamma a_i + \delta b_i, \qquad (i=0, 1, 2, \dots, n);$$

that is,

$$f(a_0', \dots, a_n', b_0', \dots, b_n') = (\alpha\delta - \beta\gamma)^u f(a_0, \dots a_n, b_0, \dots b_n).$$

From this we can see that an invariant or covariant with coefficients in terms of determinants like $\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$ also satisfies the further test and is a combinant. This same argument can be extended for three binary n-ics by regarding the elements of the matrix as coordinates of points in a plane.

III. *Operators: Their Application to Invariants and Covariants; Combinants Formed from Their Leading Coefficients by the Use of Operators.*

We shall be making constant use of two differential operators:†

$$\Omega = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + \dots + pa_{p-1} \frac{\partial}{\partial a_p}$$

and

$$O = pa_1 \frac{\partial}{\partial a_0} + (p-1) a_2 \frac{\partial}{\partial a_1} + \dots + a_p \frac{\partial}{\partial a_{p-1}}.$$

For an invariant, I, of a single quantic, they are called "annihilators," since $\Omega I = 0$ and $OI = 0$; that is, the action of either operator on I causes it to vanish.

On the other hand, these two operators enable us to form all the coefficients of a covariant of a single quantic from its end coefficient. If the end coefficient is that of the first term, we call it the *source* or *leading coefficient.* If we represent this source by C_0, then

$$\Omega C_0 = 0.$$

This is the first requirement for a source; namely, that it must vanish under Ω. However, the operator O behaves much differently, for $OC_0 = C_1$, $OC_1 = \frac{1}{2} C_2$,

* *Math. Ann.*, Vol. V.

† Cf. Elliott, "Algebra of Quantics," p. 112 *et seq.*

$\ldots, OC_n=0$, where C_n is the last coefficient. In general, we may write

$$C_k=\frac{1}{k!}(O)^k C_0,$$

where C_k is the coefficient of the $(k+1)$-th term.

For the covariants and invariants of a system of quantics of the same order p, we may apply the same arguments as before if we use for Ω and O the sum of the operators formed for each of the quantics; thus,

$$\begin{aligned}\Omega=&\left(a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+3a_2\frac{\partial}{\partial a_3}+\ldots+pa_{p-1}\frac{\partial}{\partial a_p}\right)\\&+\left(b_0\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial b_2}+\ldots+pb_{p-1}\frac{\partial}{\partial b_p}\right)\\&+\left(c_0\frac{\partial}{\partial c_1}+2c_1\frac{\partial}{\partial c_2}+\ldots+pc_{p-1}\frac{\partial}{\partial c_p}\right)+\ldots\end{aligned}$$

and

$$\begin{aligned}O=&\left(pa_1\frac{\partial}{\partial a_0}+(p-1)a_2\frac{\partial}{\partial a_1}+\ldots+a_p\frac{\partial}{\partial a_{p-1}}\right)\\&+\left(pb_1\frac{\partial}{\partial b_0}+(p-1)b_2\frac{\partial}{\partial b_1}+\ldots+b_p\frac{\partial}{\partial b_{p-1}}\right)\\&+\left(pc_1\frac{\partial}{\partial c_0}+(p-1)c_2\frac{\partial}{\partial c_1}+\ldots+c_p\frac{\partial}{\partial c_{p-1}}\right)+\ldots.\end{aligned}$$

A combinant of several forms of the same order has already been defined as an invariant or covariant of this system of forms. Hence, if we have a proper leading coefficient, we may determine the others according to the method just outlined. In the last section we also showed that an n-rowed determinant from the matrix of the coefficients of our system of n forms would be a proper coefficient for a combinant. Hence, if we choose such a combination of determinants of one weight* as will vanish under Ω, it will be a proper leading coefficient, and the remaining coefficients may be easily calculated.

IV. *The effect of operating with O on a determinant is to replace an entire column of the determinant by a new column.*

Take an n-rowed determinant of the type we are considering, say

$$\Delta=\begin{vmatrix}a_0 & a_1 & a_2 & \ldots & a_{n-1}\\ b_0 & b_1 & b_2 & \ldots & b_{n-1}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ n_0 & n_1 & n_2 & \ldots & n_{n-1}\end{vmatrix}.$$

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* We define the weight of the determinant $\begin{vmatrix}a_i & a_j\\ b_i & b_j\end{vmatrix}$, which we write symbolically $|ij|$, as $i+j-1$; and the weight of $\begin{vmatrix}a_i & a_j & a_k\\ b_i & b_j & b_k\\ c_i & c_j & c_k\end{vmatrix}$, written $|ijk|$, as $i+j+k-3$. This method is arbitrarily chosen so that the determinant of lowest weight shall be of weight zero.

If A_i is the minor of a_i, and B_i is the minor of b_i, etc., we have the following expansion of Δ in terms of the elements of the r-th column and its minors:

$$\Delta=(-1)^r(a_r A_r+b_r B_r+c_r C_r+\ldots.+n_r N_r).$$

Now the terms of O that would operate on the elements of the r-th column are

$$(n-r)\left(a_{r+1}\frac{\partial}{\partial a_r}+b_{r+1}\frac{\partial}{\partial b_r}+\ldots.\right),$$

and when we operate with them we get

$$(n-r)(-1)^r(a_{r+1}A_r+b_{r+1}B_r+\ldots.+n_{r+1}N_r),$$

where the only effect on the determinant has been to replace the entire r-th column by a new one. We have considered *any* column; hence the total effect of the operator O is to form a sum of such determinants in which one column of the original determinant is replaced by a new column.

In our notation for these determinants, the elements i, j, k of the symbol stand for the common subscripts of the letters in the column they represent. For convenient writing we consider the summed operator O as

$$O=pa_1\frac{\partial}{\partial a_0}+(p-1)a_2\frac{\partial}{\partial a_1}+\ldots.,$$

and consider the effect on subscripts rather than on letters, really counting the symbols as being subscripts of the single letter a.

Example: For three cubics,

$$O=3a_1\frac{\partial}{\partial a_0}+2a_2\frac{\partial}{\partial a_1}+a_3\frac{\partial}{\partial a_2},$$

and the effect of operating on $|012|$, say, becomes

$$O|012|=3|112|+2|022|+|013|,$$

where the symbols $|112|$ and $|022|$ vanish because they represent determinants with two columns identical.

V. *Notation for Linear Combinants and the Syzygies.*

In our actual practice we have been led to adopt the following uniform notation for the linear combinants of systems of two and three binary forms and the syzygies of the second degree.

For two binary forms $(\alpha t)^n$ and $(\beta t)^n$, the linear combinants are named:

$$a=(at)^{2(n-1)}=(\text{symbolically})\ |\alpha\beta|\,(\alpha t)^{n-1}(\beta t)^{n-1},$$
$$b=(bt)^{2(n-3)}=(\text{symbolically})\ |\alpha\beta|^3(\alpha t)^{n-3}(\beta t)^{n-3},$$
$$c=(ct)^{2(n-4)},\ \text{etc.}$$

For the linear combinants of three binary forms, $(\alpha t)^n$, $(\beta t)^n$, $(\gamma t)^n$, we assign the following names:

$$A=(At)^{3n-6},\ B=(Bt)^{3n-10},\ C=(Ct)^{3n-12},\ D=(Dt)^{3n-14},\ E=(Et)^{3n-16},\ \text{etc.},$$

where the ordinary symbolic notation is too cumbersome to note here for the general case.

The syzygies in both cases have been numbered with Roman numerals, that of the lowest weight and highest order being called I, the numbers being used only when the syzygies actually occur. In case there are two or more syzygies of the same order in any set of forms, they are distinguished by an Arabic numeral suffix; thus, IV–1, IV–2, IV–3.

VI. *Computation of the Linear Combinants of a System of Two Binary Quartics $(\alpha t)^4$ and $(\beta t)^4$, Written with Binomial Coefficients.*

For this case the operators are

$$\Omega= a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+3a_2\frac{\partial}{\partial a_3}+4a_3\frac{\partial}{\partial a_4},$$

$$O=4a_1\frac{\partial}{\partial a_0}+3a_2\frac{\partial}{\partial a_1}+2a_3\frac{\partial}{\partial a_2}+a_4\frac{\partial}{\partial a_3}.$$

The only determinant of weight zero that we have is $|01|$. $\Omega|01|=0$; hence it may be used as a leading coefficient. Call it a_0; then

$$\begin{aligned}
a_0&= |01|,\\
a_1&= Oa_0=3|02|,\\
a_2&=\frac{1}{2}Oa_1=3|03|+6|12|,\\
a_3&=\frac{1}{3}Oa_2=|04|+8|13|,\\
a_4&=\frac{1}{4}Oa_3=3|14|+6|23|,\\
a_5&=\frac{1}{5}Oa_4=3|24|,\\
a_6&=\frac{1}{6}Oa_5=|34|,\\
a_7&=\frac{1}{7}Oa_6=0.
\end{aligned}$$

Since a_7 is zero, we evidently have a form of order 6 in the variables, whence

$$a=a_0t^6+a_1t^5+a_2t^4+a_3t^3+a_4t^2+a_5t+a_6,$$

where the coefficients are computed as above.

Consider now determinants of weight 1. There is but one, $|02|$. $\Omega|02|\neq 0$; therefore it cannot be a leading coefficient of a combinant.

We have two determinants, $|03|$ and $|12|$, of weight 2. Let us find a linear combination of them, say $\lambda_1|03|+\lambda_2|12|$, such that $\Omega(\lambda_1|03|+\lambda_2|12|)=0$.

$$\Omega(\lambda_1|03|+\lambda_2|12|)=3\lambda_1|02|+\lambda_2|02|.$$

Evidently the condition that this vanish is $3\lambda_1+\lambda_2=0$. This occurs most simply when $\lambda_1=1$ and $\lambda_2=-3$; whence $|03|-3|12|$ is a proper leading coefficient of weight 2. Call it b_0; then we have

$$\begin{aligned}
b_0&= && |03|-3|12|,\\
b_1&= Ob_0&&=|04|-2|13|,\\
b_2&=\frac{1}{2}Ob_1&&=|14|-3|23|,\\
b_3&=\frac{1}{3}Ob_2&&=0.
\end{aligned}$$

Hence our second combinant is of order 2 and is

$$b=b_0t^2+b_1t+b_2.$$

The determinants of next weight are $|04|$ and $|13|$. A linear combination of these, say $\lambda_1|04|+\lambda_2|13|$, will not vanish under Ω unless $\lambda_1=\lambda_2=0$; hence we have no leading coefficient of weight 3. In this same way we can show that there are no other leading coefficients of greater weight.

VII. *Computation of the Linear Combinants of a System of Three Binary Quartics, $(\alpha t)^4$, $(\beta t)^4$, $(\gamma t)^4$, Written with Binomial Coefficients.*

The operators are the same as in Section VI.

The determinant of weight zero is $|012|$ and vanishes under Ω; hence it may form a leading coefficient. The coefficients are thus:

$$\begin{aligned}
A_0&= && |012|,\\
A_1&= OA_0&&=2|013|,\\
A_2&=\frac{1}{2}OA_1&&=|014|+3|023|,\\
A_3&=\frac{1}{3}OA_2&&=2|024|+4|123|,\\
A_4&=\frac{1}{4}OA_3&&=|034|+3|124|,\\
A_5&=\frac{1}{5}OA_4&&=2|134|,\\
A_6&=\frac{1}{6}OA_5&&=|234|,\\
A_7&=\frac{1}{7}OA_6&&=0.
\end{aligned}$$

From these we can see that we have a combinant of order 6. It is

$$A = A_0 t^6 + A_1 t^5 + A_2 t^4 + A_3 t^3 + A_4 t^2 + A_5 t + A_6 .$$

There is only one determinant of weight 1; it is $|013|$. $\Omega |013| \neq 0$; therefore we can have no combinant beginning with a coefficient of weight 1.

Of weight 2 we have the determinants $|014|$ and $|023|$. Let us find a linear combination of these two, say $\lambda_1 |014| + \lambda_2 |023|$, which will vanish under Ω.

$$\Omega (\lambda_1 |014| + \lambda_2 |023|) = 4\lambda_1 |013| + 2\lambda_2 |013|,$$

which will vanish if $\lambda_1 = 1$ and $\lambda_2 = -2$, whence $|014| - 2|023|$ is a proper leading coefficient of weight 2. From it we obtain, as before, the coefficients of the combinant

$$B = (|014| - 2|023|) t^2 + (|024| - 8|123|) t + |034| - 2|124| .$$

The determinants of weight greater than 2 will not form linear combinations which will vanish under Ω; hence there are no further linear combinants for three quartics.

VIII. *Determinant Identities.*

There exist among the coefficients of the linear combinants certain quadratic identities. The types we shall use are:

(1). Like $|01||23| + |02||31| + |03||12| = 0$, where one column is fixed and the others are cyclically permuted.

(2). Like $|012||034| + |013||042| + |014||023| = 0$, which is an extended form of type (1). It is carried over from the binary to the ternary domain by prefixing a common column to each determinant and extending the columns from two to three rows.

(3). Like $|123||456| - |234||156| + |341||256| - |412||356| = 0$, where two columns are fixed and the others are permuted cyclically.

All three types may be shown to vanish identically by expansion. However, it is interesting to see how they may be explained geometrically. The first is merely the linear condition existing between four collinear points. It is the only identity that concerns us in the binary domain. From it we get, by the Clebsch "principle of transference," the identity of type (2) for the ternary domain. We may consider 1, 2, 3, 4 as the names of four points in a plane. If we connect them with an arbitrary point 0, we have in the identity the coincidence relation of the lines $\overline{01}, \overline{02}, \overline{03}, \overline{04}$.

Type (3) is indigenous with the ternary domain. Starting with any six numbers, they may be arranged to satisfy this identity in fifteen different ways, but these represent only five independent linear relations among the six numbers.

IX. *Syzygies of Degree 2 Connecting the Linear Combinants.*

There are certain identities of degree 2 existing among the transvectants of the linear combinants of any system of quantics. We shall now show how to determine these identities.

Let us consider, for instance, the linear combinants of the three quartics as given in Section VII. They are A, of order 6 and weight zero,* and B, of order 2 and weight 2.

The transvectants of second degree arising from these two forms—counting their squares and product as transvectants—are given by the table:

	Transvectants.	Order.	Weight.
I.	A^2	12	0
II.	$\|A, A\|^2$, AB	8	2
III.	$\|A, B\|$	6	3
IV.	$\|A, A\|^4$, B^2, $\|A, B\|^2$	4	4
V.	$\|A, A\|^6$, $\|B, B\|^2$	0	6

The determinant identities are all of type (2), formed from the numbers 0, 1, 2, 3, 4. We have the following identities, one term being given for the whole identity:

$|012|\,|034|$ of weight 4, $\quad |304|\,|312|$ of weight 7,
$|104|\,|123|$ of weight 5, $\quad |403|\,|412|$ of weight 8.
$|204|\,|213|$ of weight 6,

The syzygies depend upon the identities of the same weight to such an extent that if we do not have any identity of the desired weight, we can not have any syzygy. As the weight of the lowest identity is 4, the transvectants of the rows I, II and III can not give rise to any syzygies. Let us now consider the row IV.

$|A, A|^4$, the fourth transvectant of A on itself, is obtained by taking the fourth polar of A and forming its bilinear invariant. Since $A = (At)^6$ is written without the binomial coefficients, we have this fourth polar as

$$(A_0 t_1^2 + 2A_1 t_1 t_2 + A_2 t_2^2) \cdot T_1^4 + (A_1 t_1^2 + A_2 t_1 t_2 + A_3 t_2^2) \cdot 4T_1^3 T_2$$
$$+ (A_2 t_1^2 + A_3 t_1 t_2 + A_4 t_2^2) \cdot 6T_1^2 T_2^2 + (A_3 t_1^2 + A_4 t_1 t_2 + A_5 t_2^2) \cdot 4T_1 T_2^3$$
$$+ (A_4 t_1^2 + A_5 t_1 t_2 + A_6 t_2^2) \cdot T_2^4,$$

whence

$$|A, A|^4 = 2(A_0 A_4 - 4A_1 A_3 + 3A_2 A_2) t_1^4 + \ldots\ldots$$

The second form, B^2, is merely the square of $(Bt)^2$, whence

$$B^2 = |\,(Bt)^2\,|^2 = B_0^2 t_1^4 + \ldots\ldots$$

* In the succeeding discussion we shall repeatedly refer to the weight of a linear combinant or syzygy as identical with the weight of the leading coefficient.

The third form, $|A, B|^2$, is the second transvectant of A on B. It is then the bilinear invariant of the second polar of A and the second polar of B.

The second polar of A is

$$\begin{aligned}&T_1^2(A_0 t_1^4+4A_1 t_1^3 t_2+6A_2 t_1^2 t_2^2+4A_3 t_1 t_2^3+A_4 t_2^4)\\ &\qquad+2T_1 T_2(A_1 t_1^4+4A_2 t_1^3 t_2+6A_3 t_1^2 t_2^2+4A_4 t_1 t_2^3+A_5 t_2^4)\\ &\qquad+T_2^2(A_2 t_1^4+4A_3 t_1^3 t_2+6A_4 t_1^2 t_2^2+4A_5 t_1 t_2^3+A_6 t_2^4),\end{aligned}$$

and the second polar of B is

$$B_0 T_1^2+2B_1 T_1 T_2+B_2 T_2^2,$$

whence

$$|A, B|^2=(A_0 B_2-2A_1 B_1+A_2 B_0)t^4+\ldots\ldots$$

We have in each case written only the leading coefficients of the transvectants, since these determine all the others in such a way that whatever linear relations connect the leading coefficients of these transvectants will also hold for any other set of coefficients. Now in terms of $(At)^6$ and $(Bt)^2$ expressed with binomial coefficients, the A's and B's have the following values:

$$\begin{aligned}A_0&=|012|, & B_0&=|014|-2|023|,\\ A_1&=\frac{1}{3}|013|, & B_1&=\frac{1}{2}(|024|-8|123|),\\ A_2&=\frac{1}{15}(|014|+3|023|), & B_2&=|034|-2|124|,\\ A_3&=\frac{1}{10}(|024|+2|123|), & &\\ A_4&=\frac{1}{15}(|034|+3|124|), & &\\ &\ldots\ldots\ldots\ldots\ldots\ldots & &\end{aligned}$$

Using these values, we set down the actual values of the leading coefficients of $|A, A|^4$, B^2, and $|A, B|^2$.

	$\lvert 012\rvert\,\lvert 124\rvert$	$\lvert 012\rvert\,\lvert 034\rvert$	$\lvert 013\rvert\,\lvert 123\rvert$	$\lvert 013\rvert\,\lvert 024\rvert$	$\lvert 023\rvert^2$	$\lvert 023\rvert\,\lvert 014\rvert$	$\lvert 014\rvert^2$
$\lvert A, A\rvert^4$	2/5	2/15	—8/15	—4/15	6/25	4/25	2/75
B^2	....	....	....	....	4	—4	1
$\lvert A, B\rvert^2$	—2	1	8/3	—1/3	—2/5	1/15	1/15
	(1)	I	(2)	I	(3)	I	(4)

The determinants marked with the Roman I are connected by the identity

$$|012|\,|034|+|013|\,|024|+|014|\,|023|=0.$$

If now the entire first row be multiplied by λ_1, the second by λ_2, and the third by λ_3, the λ's may, as we shall show, be so determined that

$$\lambda_1 | A, A |^4 + \lambda_2 B^2 + \lambda_3 | A, B |^2 = 0.$$

To accomplish this, we must satisfy the equations formed by equating to zero the sum of the coefficients of each of the columns marked with an Arabic numeral. Thus we have the four simultaneous equations in λ_1:

$$\begin{array}{ll} (1) & \lambda_1 \qquad\qquad -5\lambda_3 = 0, \\ (2) & -\lambda_1 \qquad\qquad +5\lambda_3 = 0, \\ (3) & 3\lambda_1 + 50\lambda_2 - 5\lambda_3 = 0, \\ (4) & 2\lambda_1 + 75\lambda_2 + 5\lambda_3 = 0, \end{array}$$

from which, since not all the equations are independent, we can determine the ratios of the λ's. We have, for the simplest integral values,

$$\lambda_1 = 25, \quad \lambda_2 = -1, \quad \lambda_3 = 5.$$

These values will make each of the four columns from which the equations were obtained vanish. For the columns marked I their behavior is much different. When they are substituted in these latter columns, they give in each case the absolute value $\frac{25}{3}$. Hence, for these three columns taken together, we have the value

$$\frac{25}{3} \left(|012|\,|034| + |013|\,|042| + |014|\,|023| \right),$$

which vanishes independently of the λ's, since the quantity in the parentheses is one of the determinant identities of the second type mentioned in the last section.

We have, therefore, for our syzygy,

$$25 | A, A |^4 - B^2 + 5 | A, B |^2 = 0.$$

This is an expression of order 4 in the variables and will therefore, when expanded, have five terms. The weights of these terms are 4, 5, 6, 7 and 8, respectively. We have already shown (p. 254) that there is just one determinant identity of each of these weights, so that there are just enough identities to account for all the terms of the syzygy, and there are none left over for the formation of any other syzygies; hence we can have no other syzygies connecting the linear combinants of three quartics. There is, then, a unique syzygy of degree 2 connecting the linear combinants of three binary quartics, itself a quartic syzygy.

X. *Tables of Linear Combinants of Systems of Binary Forms of the Same Order, with the Syzygies of the Second Degree Connecting Them.*

Following the methods just briefly illustrated, we have computed the linear combinants for systems of two and three binary forms, up to and including the case of binary septimics, and have either computed or enumerated the syzygies of the second degree connecting them. The results of these computations are presented in the following set of tables.

The works of a number of investigators who have conducted researches along these lines previous to this paper have been consulted in the preparation of this work for corroborative evidence. Those papers of particular value in this instance are one by Stroh* in which he gives a splendid enumeration of the linear combinants of systems of two, three and four binary forms, and two by Berzolari† in which are given the linear and quadratic combinants of systems of two and three quartics and quintics. Both of these authors have obtained their results through considerations other than those which are presented here.

Linear Combinants of Two Binary Forms, $(\alpha t)^n$ and $(\beta t)^n$, and the Syzygies of the Second Degree Connecting Them.

Two Quadratics. Linear combinant:

$$a = |01|t^2 + |02|t + |12|.$$

Two Cubics. Linear combinants:

$$a = |01|t^4 + 2|02|t^3 + (|03| + 3|12|)t^2 + 2|13|t + |23|,$$
$$b = |03| - 3|12|.$$

Syzygy:

I. $6|a, a|^4 - b^2 = 0.$

Two Quartics. Linear combinants:

$$a = |01|t^6 + 3|02|t^5 + (3|03| + 6|12|)t^4 + (|04| + 8|13|)t^3$$
$$+ (3|14| + 6|23|)t^2 + 3|24|t + |34|,$$
$$b = (|03| - 3|12|)t^2 + (|04| - 2|13|)t + (|14| - 3|23|).$$

Syzygy:

I. $|a, a|^4 - |a, b|^2 - b^2 = 0.$

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* E. Stroh, "Zur Theorie der Combinanten," *Math. Ann.*, Vol. XXII (1893), p. 404.

† L. Berzolari, "Combinanti dei sistemi lineari di quintiche binarie," *Cir. Mt. Rd. Palermo*, Vol. VII (1893), pp. 5–18; "Binary Combinants Associated with Curves of the Fourth Order," *Annali di Matematica*, Vol. XX (1883), pp. 101–105.

Two Quintics. Linear combinants:

$$a=|01|t^8+4|02|t^7+2(3|03|+5|12|)t^6+4(|04|+5|13|)t^5$$
$$+(|05|+15|14|+20|23|)t^4+4(|15|+5|24|)t^3$$
$$+2(3|25|+5|34|)t^2+4|35|t+|45|,$$

$$b=(|03|-3|12|)t^4+2(|04|-2|13|)t^3+(|05|+|14|-8|23|)t^2$$
$$+2(|15|-2|24|)t+|25|-3|34|,$$

$$d=|05|-5|14|+10|23|.$$

Syzygies (formed with $210a$, $30b$ and $14d$):

I. $2|a,a|^4-15b^2+8|a,b|^2+2ad=0.$

II. $2|a,a|^6-50|b,b|^2-15|a,b|^4-35bd=0.$

III. $|a,a|^8+35|b,b|^4-30d^2=0.$

Two Sextics. Linear combinants:

$$a=|01|t^{10}+5|02|t^9+5(2|03|+3|12|)t^8+10(|04|+4|13|)t^7$$
$$+5(|05|+9|14|+10|23|)t^6+(|06|+24|15|+75|24|)t^5$$
$$+5(|16|+9|25|+10|34|)t^4+10(|26|+4|35|)t^3$$
$$+5(2|36|+3|45|)t^2+5|46|t+|56|,$$

$$b=(|03|-3|12|)t^6+3(|04|-2|13|)t^5+3(|05|-5|23|)t^4$$
$$+(|06|+6|15|-15|24|)t^3+3(|16|-5|34|)t^2$$
$$+3(|26|-2|35|)t+(|36|-3|45|),$$

$$d=(|05|-5|14|+10|23|)t^2+(|06|-4|15|+5|24|)t$$
$$+(|16|-5|25|+10|34|).$$

Syzygies:

I. $140b^2-945|a,a|^4+630|a,b|^2-81|a,d|=0.$

II. $14|a,b|^5-3|b,d|=0.$

III. $252|a,a|^6-140|b,b|^2+210|a,b|^4-54|a,d|^2+33|b,d|=0.$

IV. $243d^2-1890|a,a|^8-1400|b,b|^4+840|a,b|^6+540|b,d|^2=0.$

V. $126|a,a|^{10}-280|b,b|^6+27|d,d|^2=0.$

Two Septimics. Linear combinants:

$$a=|01|t^{12}+6|02|t^{11}+3(5|03|+7|12|)t^{10}+10(2|04|+7|13|)t^9$$
$$+15(|05|+7|14|+7|23|)t^8+6(|06|+14|15|+35|24|)t^7$$
$$+(|07|+35|16|+189|25|+175|34|)t^6$$
$$+6(|17|+14|26|+35|35|)t^5+15(|27|+7|36|+7|45|)t^4$$
$$+10(2|37|+7|46|)t^3+3(5|47|+7|56|)t^2+6|57|t+|67|,$$

$$b=(|03|-3|12|)t^8+4(|04|-2|13|)t^7+2(3|05|-|14|-12|23|)t^6$$
$$+4(|06|+3|15|-9|24|)t^5+(|07|+13|16|-9|25|-45|34|)t^4$$
$$+4(|17|+3|26|-9|35|)t^3+2(3|27|-|36|-12|45|)t^2$$
$$+4(|37|-2|46|)t+(|47|-3|56|),$$

$$d=(|05|-5|14|+10|23|)t^4+2(|06|-4|15|+5|24|)t^3$$
$$+(|07|-|16|-9|25|+25|34|)t^2+2(|17|-4|26|+5|35|)t$$
$$+(|27|-5|36|+10|45|),$$

$$f=|07|-7|16|+21|25|-35|34|.$$

Syzygies (formed with $924a$, $70b$, $6d$ and f):

I. $15|a, a|^4-378b^2-144|a, b|^2+220ad=0.$

II. $5|a, a|^6-504|b, b|^2+45|a, b|^4-220|a, d|^2-1155bd+165af=0.$

III. $3|a, b|^5-5|a, d|^3-98|b, d|=0.$

IV–1 and IV–2. Terms involving the following transvectants:

$$|a, a|^8,\ |b, b|^4,\ d^2,\ |a, b|^6,\ |a, c|^4,\ |b, d|^2,\ bf,\ |a, b|^7,\ |b, d|^3.$$

V–1 and V–2. Terms involving the following transvectants:

$$|a, a|^{10},\ |b, b|^6,\ |d, d|^2,\ |a, b|^8,\ |b, d|^4,\ df.$$

VI. $5|a, a|^{12}+1386|b, b|^8+59, 290|d, d|^4-457, 380f^2=0$

Linear Combinants of Three Binary Forms, $(\alpha t)^n$, $(\beta t)^n$, $(\gamma t)^n$, and the Syzygies of the Second Degree Connecting Them.

Three Cubics. Linear combinant:

$$A=|012|t^3+|013|t^2+|023|t+|123|.$$

Three Quartics. Linear combinants:

$$A=|012|t^6+2|013|t^5+(|014|+3|023|)t^4+(2|024|+4|123|)t^3$$
$$+(|034|+3|124|)t^2+2|134|t+|234|,$$

$$B=(|014|-2|023|)t^2+(|024|-8|123|)t+(|034|-2|124|).$$

Syzygy:

I. $25|A, A|^4-B^2+5|A, B|^2=0.$

Three Quintics. Linear combinants:

$$A=|012|t^9+3|013|t^8+(3|014|+6|023|)t^7,$$
$$+(|015|+8|024|+10|123|)t^6+(3|025|+6|034|+15|124|)t^5$$
$$+(3|035|+6|125|+15|134|)t^4+(|045|+8|135|+10|234|)t^3$$
$$+(3|145|+6|235|)t^2+3|245|t+|345|,$$

$$B=(|014|-2|023|)t^5+(|015|-10|123|)t^4+(2|025|-10|124|)t^3$$
$$+(2|035|-10|134|)t^2+(|045|-10|234|)t+(|145|-2|235|),$$

$$C=(2|015|-5|024|+20|123|)t^3+(3|025|-15|034|+15|124|)t^2$$
$$-(3|035|-15|125|+15|134|)t-(2|045|-5|135|+20|234|).$$

Syzygies (formed with $84A$, $5B$ and C):

I. $25|A, A|^4-294B^2-100|A, C|=0.$

II. $5|A, B|^3-8BC=0.$

III. $5|A, A|^6+294|B, B|^2+192C^2+40|A, C|^3=0.$

IV. $|A, B|^5-16|B, C|^2=0.$

V. $5|A, A|^8-196|B, B|^4+112|C, C|^2=0.$

Three Sextics. Linear combinants:

$$\begin{aligned}
A=&\,|012|t^{12}+4|013|t^{11}+(6|014|+10|023|)t^{10}\\
&+(4|015|+20|024|+20|123|)t^9\\
&+(|016|+15|025|+20|034|+45|124|)t^8\\
&+(4|026|+20|035|+36|125|+60|134|)t^7\\
&+(6|036|+10|045|+10|126|+64|135|+50|234|)t^6\\
&+(4|046|+20|136|+36|145|+60|235|)t^5\\
&+(|056|+15|146|+20|236|+45|245|)t^4\\
&+(4|156|+20|246|+20|345|)t^3\\
&+(6|256|+10|346|)t^2+4|356|t+|456|,
\end{aligned}$$

$$\begin{aligned}
B=&\,(|014|-2|023|)t^8+(2|015|-|024|-12|123|)t^7\\
&+|(016|+4|025|-2|034|-21|124|)t^6\\
&+(3|026|+4|035|-6|125|-32|134|t^5\\
&+(4|036|+3|045|+3|126|-16|135|-40|234|)t^4\\
&+(3|046|+4|136|-6|145|-32|235|)t^3\\
&+(|056|+4|146|-2|236|-21|245|)t^2\\
&+(2|156|-|246|-12|345|)t+|256|-2|346|,
\end{aligned}$$

$$\begin{aligned}
C=&\,(2|015|-5|024|+20|123|)t^6+(2|016|-20|034+30|124|)t^5\\
&+(5|026|-20|035|+30|125|)t^4-(20|045|-20|126|)t^3\\
&-(5|046|-20|136|+30|145)t^2-(2|056|-20|236|+30|245|)t\\
&-(2|156|-5|246|+20|345|),
\end{aligned}$$

$$\begin{aligned}
D=&\,(|016|-3|025|+5|034|)t^4\\
&+(2|026|-2|035|-18|125|+30|134|)t^3\\
&+(3|036|-3|045|-3|126|-12|135|+75|234|)t^2\\
&+(2|046|-2|136|-18|145|+30|235|)t\\
&+(|056|-3|146|+5|236|),
\end{aligned}$$

$$E=|036|-3|045|-3|126|+6|135|-15|234|.$$

Syzygies (formed with $6930A$, $560B$, $3C$, $2D$ and E):

I. $64|A, A|^4-405B^2-72|A, B|^2-7392|A, C|+3520AD=0,$

II. $|A, B|^3-14|A, C|^2-42BC-16|A, D|=0,$

III–1 and III–2. Involving the following transvectants:

$|A, A|^6$, $|B, B|^2$, C^2, $|A, B|^4$, $|A, C|^3$, $|B, C|$, AF, $|A, D|^2$, AD.

IV–1 and IV–2. Involving the following transvectants:

$|A, B|^5$, $|A, C|^4$, $|B, C|^2$, $|A, D|^3$ $|B, D|$, CD.

V–1, V–2 and V–3. Involving the following transvectants:

$|A, A|^8$, $|B, B|^4$, $|C, C|^2$, $|A, B|^6$, $|A, C|^5$, $|B, C|^3$, $|A, D|^4$, BF, $|C, D|^2$, D^2, $|C, D|$.

VI–1 and VI–2. Involving the following transvectants:

$|A, B|^7$, $|A, C|^6$, $|B, C|^4$, $|B, D|^3$, CF, $|C, D|^2$.

VII–1, VII–2 and VII–3. Involving the following transvectants:

$|A, A|^{10}$, $|B, B|^6$, $|C, C|^4$, $|A, B|^8$, $|B, C|^5$, $|B, D|^4$, $|C, D|^3$, DF, $|D, D|^2$.

VIII. $80|B, C|^6-|C, D|^4=0$.

IX. Involving $|A, A|^{12}$, $|B, B|^8$, $|C, C|^6$, $|D, D|^4$, F^2.

Three Septimics. Linear combinants:

$$
\begin{aligned}
A= & |012|t^{15}+5|013|t^{14}+(10|014|+15|023|)t^{13} \\
& +(10|015|+40|024|+35|123|)t^{12} \\
& +(5|016|+45|025|+50|034|+105|124|)t^{11} \\
& +(|017|+24|026|+75|035|+126|125|+175|134|)t^{10} \\
& +(5|027|+45|036|+50|045|+70|126|+280|135|+175|234|)t^{9} \\
& +(10|037|+40|046|+15|127|+175|136|+210|145|+315|235|)t^{8} \\
& +(10|047|+40|137|+15|056|+175|146|+210|236|+315|245|)t^{7} \\
& +(5|057|+45|147|+50|237|+70|156|+280|246|+175|345|)t^{6} \\
& +(|067|+24|157|+75|247|+126|256|+175|346|)t^{5} \\
& +(5|167|+45|257|+50|347|+105|356|)t^{4} \\
& +(10|267|+40|357|+35|456|)t^{3} \\
& +(10|367|+15|457|)t^{2}+5|467|t+|567|,
\end{aligned}
$$

$$
\begin{aligned}
B= & (|014|-2|023|)t^{11}+(3|015|-2|024|-14|123|)t^{10} \\
& +(3|016|+6|025|-5|034|-35|124|)t^{9} \\
& +(|017|+10|026|+5|035|-21|125|-70|134|)t^{8} \\
& +(4|027|+15|036|+5|045|+7|126|-70|135|-105|234|)t^{7} \\
& +(7|037|+14|046|+7|127|-49|145|-147|235|)t^{6} \\
& +(7|047|+14|137|+7|056|-49|236|-147|245|)t^{5} \\
& +(4|057|+15|147|+5|237|+7|156|-70|246|-105|345|)t^{4} \\
& +(|067|+10|157|+5|247|-21|256|-70|346|)t^{3} \\
& +(3|167|+6|257|-5|347|-35|356|)t^{2} \\
& +(3|267|-2|357|-14|456|)t+(|367|-2|457|),
\end{aligned}
$$

$$C=(2|015|-5|024|+20|123|)t^9+(4|016|-3|025|-25|034|+45|124|)t^8$$
$$+(2|017|+9|026|-45|035|+57|125|+25|134|)t^7$$
$$+(7|027|-15|036|-60|045|+59|126|+15|135|+50|234|)t^6$$
$$+(5|037|-45|046|+27|127|+55|136|-90|145|+60|235|)t^5$$
$$-(5|047|-45|137|+27|056|+55|146|-90|236|+60|245|)t^4$$
$$-(7|057|-15|147|-60|237|+59|156|+15|246|+50|345|)t^3$$
$$-(2|067|+9|157|-45|247|+57|256|+25|346|)t^2$$
$$-(4|167|-3|257|-25|347|+45|356|)t-(2|267|-5|357+20|456|),$$

$$D=(|016|-3|025|+5|034|)t^7+(|017|-21|125|+35|134|)t^6$$
$$+(3|027|-21|126|+105|234|)t^5+(5|037|-35|136|+105|235|)t^4$$
$$+(5|047|-35|146|+105|245|)t^3+(3|057|-21|156|+105|345|)t^2$$
$$+(|067|-21|256|+35|346|)t+(|167|-3|257|+5|347|),$$

$$E=(2|017|-7|026|+7|035|+21|125|-35|134|)t^5$$
$$+(5|027|-21|036|+28|045|-7|126|+49|135|-210|234|)t^4$$
$$+(2|037|-14|046|+14|127|-42|136|+196|145|-168|235|)t^3$$
$$-(2|047|-14|137|+14|056|-42|146|+196|236|-168|245|)t^2$$
$$-(5|057|-21|147|+28|237|-7|156|+49|246|-210|345|)t$$
$$-(2|067|-7|157|+7|247|+21|256|-35|346|),$$

$$F=(|036|-3|045|-3|126|+6|135|-15|234|)t^3$$
$$+(|037|-2|046|-3|127|+4|136|+3|145|-9|235|)t^2$$
$$+(|047|-2|137|-3|056|+4|146|+3|256|-9|245|)t$$
$$+(|147|-3|237|-3|156|+6|246|-15|345|).$$

Syzygies. In the following table we have enumerated the syzygies and indicated the transvectants which they will involve:

I.	$\lvert A, A\rvert^4$, B^2, $\lvert A, B\rvert^2$, $\lvert A, C\rvert$, AD.
II.	$\lvert A, B\rvert^3$, $\lvert A, C\rvert^2$, $\lvert A, D\rvert$, AE, BC.
III–1 and 2.	$\lvert A, A\rvert^6$, $\lvert B, B\rvert^2$, C^2, $\lvert A, B\rvert^4$, $\lvert A, C\rvert^3$, $\lvert A, D\rvert^2$, $\lvert A, E\rvert$, $\lvert B, C\rvert$, AF, BD.
IV–1 to 3.	$\lvert A, B\rvert^5$, $\lvert A, C\rvert^4$, $\lvert A, D\rvert^3$, $\lvert A, E\rvert^2$, $\lvert A, F\rvert$, $\lvert B, C\rvert^2$, $\lvert B, D\rvert$, BE, CD.
V–1 to 4.	$\lvert A, A\rvert^8$, $\lvert B, B\rvert^4$, $\lvert C, C\rvert^2$, D^2 $\lvert A, B\rvert^6$, $\lvert A, C\rvert^5$, $\lvert A, D\rvert^4$, $\lvert A, E\rvert^3$, $\lvert A, F\rvert^2$, $\lvert B, C\rvert^3$, $\lvert B, D\rvert^2$, $\lvert B, E\rvert$, BF, $\lvert C, D\rvert$, CE.
VI–1 to 4.	$\lvert A, B\rvert^7$, $\lvert A, C\rvert^6$, $\lvert A, D\rvert^5$, $\lvert A, E\rvert^4$, $\lvert A, F\rvert^3$, $\lvert B, C\rvert^4$, $\lvert B, D\rvert^3$, $\lvert B, E\rvert^2$, $\lvert B, F\rvert$, $\lvert C, D\rvert^2$, $\lvert CE\rvert$, CF, DE.
VII–1 to 6.	$\lvert A, A\rvert^{10}$, $\lvert B, B\rvert^6$, $\lvert C, C\rvert^4$, $\lvert D, D\rvert^2$, E^2, $\lvert A, B\rvert^8$, $\lvert A, C\rvert^7$, $\lvert A, D\rvert^6$, $\lvert A, E\rvert^5$, $\lvert B, C\rvert^5$, $\lvert B, D\rvert^4$, $\lvert B, E\rvert^3$, $\lvert B, F\rvert^2$, $\lvert C, D\rvert^3$, $\lvert C, E\rvert^2$, $\lvert C, F\rvert$, $\lvert D, E\rvert$, DF.

VIII–1 to 5. $|A, B|^9$, $|A, C|^8$, $|A, D|^7$, $|B, C|^6$, $|B, D|^5$, $|B, E|^4$, $|B, F|^3$, $|C, D|^4$, $|C, E|^3$, $|C, F|^2$, $|D, E|^2$, $|D, F|$, EF.

IX–1 to 5. $|A, A|^{12}$, $|B, B|^8$, $|C, C|^6$, $|D, D|^4$, $|E, E|^2$, F^2, $|A, B|^{10}$, $|A, C|^9$, $|B, C|^7$, $|B, D|^6$, $|B, E|^5$, $|C, D|^5$, $|C, E|^4$, $|C, F|^3$, $|D, E|^3$, $|D, F|^2$, $|E, F|$.

X–1 to 4. $|A, B|^{11}$, $|B, C|^8$, $|B, D|^7$, $|C, D|^6$, $|C, E|^5$ $|D, E|^4$, $|D, F|^3$, $|E, F|^2$.

XI–1 to 3. $|A, A|^{14}$, $|B, B|^{10}$, $|C, C|^8$, $|D, D|^6$, $|E, E|^4$, $|F, F|^2$, $|B, C|^9$, $|C, D|^7$, $|D, E|^5$, $|E, F|^3$.

XI. *Various Methods for Checking the Computation of the Linear Combinants.*

(1). It has been shown by Clebsch* that the system of elementary covariants of a binary form with several sets of variables contains just as many linearly independent constants as the form itself. This theorem is modified by Stroh† to apply to the linear combinants of a system of p forms of order n. If the number of linearly independent constants—determinants from the matrix of the coefficients of the n-ics—be given by $\binom{n+1}{p}$, we must have the relation

$$\Sigma(\lambda_i+1)=\binom{n+1}{p},$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the orders of the linear combinants belonging to the p forms. This says, in a word, that the total number of coefficients of the linear combinants of p forms of order n must be the same as the number of different p-rowed determinants which can be formed from the matrix of the coefficients of the n-ics.

(2). The symmetry of the linear combinants gives rise to a very convenient check on the accuracy of the work. The numerical multipliers of the various determinants are the same for terms the same distance from each end of the combinant, and the determinants having the same multipliers are complements of each other. By complementary determinants with reference to a system of p binary n-ics we mean two determinants of the same number of rows and columns so related that the one can be formed from the other by reversing its columns and then subtracting each integer from n. Thus, the complement of

$$|a, b, c, \ldots, p| \text{ is } |n-p, \ldots, n-c, n-b, n-a|.$$

In each case in the present paper, all the terms in the combinant were actually computed by the methods set down, and then these checks were applied.

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* Clebsch, "Binäre Formen," p. 39. † *Math. Ann.*, Vol. XXII, p. 404.

XII. *Methods for Checking the Computation of the Syzygies.*

For the syzygies we have two methods of checking:

(1). The total number of coefficients in all the forms of second degree that can be gotten from the linear combinants of any system of p binary n-ics is the same as the total number of products of p-rowed determinants, taken two at a time, which can be formed from the $(n+1)p$ coefficients of the matrix. This check enables us to be sure that we have made no omissions when we set down the transvectants of the linear combinants.

(2). An extension of the theorem stated for the number of coefficients in the linear combinants can be made for the syzygies. It is: *The number of coefficients in all the syzygies of a system of* p *binary* n*-ics is the same as the number of independent determinant identities of degree* 2.

The actual arithmetical operations of all the involved computations of the syzygies were checked on the "Millionaire" computing machine.

XIII. *The order of a linear combinant may be expressed in terms of the weight of the leading coefficient and the common order of the binary forms of the system.*

Consider a system of two binary n-ics. Call the leading coefficient of any one of its combinants w_0, and suppose that this leading coefficient involves the determinant $|a, b|$, whence $w_0 = a + b - 1$; then the corresponding determinant in the last coefficient will be its complement, $|n-a, n-b|$, of weight $2n-a-b-1$. Since the weight increases by one in each coefficient after the first, we have for the number of coefficients, less one, in the combinant,

$$2n-a-b-1-a-b+1 = 2n-2w_0-2.$$

But the order r of the combinant is also a number one less than the number of coefficients, whence

$$r = 2n-2w_0-2.$$

This may easily be extended for the case of p binary n-ics. Suppose one of the determinants in the leading coefficient is $|a_1, a_2, \ldots, a_p|$, of weight

$$w_0 = \sum_{i=1}^{p} a_i - \sum_{s=0}^{p-1}(s+1);$$

then the corresponding determinant in the last coefficient is

$$|n-a_p, \ldots, n-a_2, n-a_1|,$$

of weight

$$p \cdot n - \sum_{i=1}^{p} a_i - \sum_{s=0}^{p-1}(s+1).$$

The number of coefficients in the combinant, less one, is then given by

$$r = p\cdot n - 2w_0 - 2\sum_{s=0}^{p-1}(s+1).$$

This is then the formula for the order of the combinant, since the order is one less than the number of coefficients in the combinant. This formula gives a convenient method of determining the order of a linear combinant when the weight of the leading coefficient is known. The term $\sum_{0}^{p-1}(s+1)$ is introduced in the expression for the weight, so that the determinant $|0, 1, 2, \ldots, (p-1)|$ shall always be of weight zero.

For the special case of a system of three binary n-ics, the above general formula becomes

$$r = 3n - 2w_0 - 6.$$

XIV. *The Enumeration of the Quadratic Syzygies Connecting the Linear Combinants of Any System of Binary Forms of the Same Order, by Means of the Number and Weight of the Possible Determinant Identities of the Second Degree.*

It has already been shown (Section XI) that there are just as many different determinants which may be formed from the matrix of the coefficients of a system of binary forms of a common order as there are coefficients in all the linear combinants of that system. By means of this relation it is quite possible to make an enumeration of the linear combinants of a system by counting the number and weights of the determinants. However, the work of actually computing the explicit forms offers too few numerical difficulties to make such an enumeration of much profit.

On the other hand, in the case of the quadratic syzygies, where the computation is often much involved, there is much to be gained through a similar enumeration. The quadratic forms which must be accounted for are of two distinct classes. First, we have the odd transvectants of a linear combinant on itself. These odd transvectants vanish identically and so will not enter into the actual formation of the syzygies. Second, we have the even transvectants of a linear combinant on itself, the transvectants of a linear combinant on another linear combinant of the same system, the products of two linear combinants and their squares. The totality of members of this second class are the constituents of the syzygies, into which they are formed by means of the relations of the various coefficients, aided by the determinant identities of the forms given in Section VIII. We have shown that the number of coefficients in the quadratic syzygies of any system is the same as the number of quadratic

determinant identities. If, now, we determine the number of determinant identities of each possible weight, we have from their totality the number of coefficients in the complete system of syzygies. We have noticed, in the formation of the syzygies whose explicit forms we have developed, that each coefficient in the syzygy requires *one* identity of the same weight as that coefficient. Hence, from the weights of the determinant identities, we can easily predict the weights of all the coefficients which can occur in the syzygies, and thus, as we shall see presently, can make a complete enumeration of the syzygies. In the succeeding sections this method of enumeration is carried out for the case of two septimics, three sextics, and three septimics.

XV. *Enumeration of the Quadratic Syzygies of the System of Two Binary Septimics.*

The linear combinants are, $a \equiv (at)^{12}$ of weight 0, $b \equiv (bt)^8$ of weight 2, $d \equiv (dt)^4$ of weight 4, and $f \equiv (ft)^0$ of weight 6. From these we may form the following quadratic products and transvectants:

Table 1.

Forms.	Ord.	Wt.
a^2	24	0
$\lvert a, a\rvert^2$, ab	20	2
$\lvert a, b\rvert$	18	3
$\lvert a, a\rvert^4$, b^2, $\lvert a, b\rvert^2$, ad	16	4
$\lvert a, b\rvert^3$, $\lvert a, d\rvert$	14	5
$\lvert a, a\rvert^6$, $\lvert b, b\rvert^2$, $\lvert a, b\rvert^4$, $\lvert a, d\rvert^2$, af, bd	12	6
$\lvert a, b\rvert^5$, $\lvert a, d\rvert^3$, $\lvert b, d\rvert$	10	7
$\lvert a, a\rvert^8$, $\lvert b, b\rvert^4$, d^2, $\lvert a, b\rvert^6$, $\lvert a, d\rvert^4$, $\lvert b, d\rvert^2$, bf	8	8
$\lvert a, b\rvert^7$, $\lvert b, d\rvert^3$	6	9
$\lvert a, a\rvert^{10}$, $\lvert b, b\rvert^6$, $\lvert d, d\rvert^2$, $\lvert a, b\rvert^8$, $\lvert b, d\rvert^4$, df	4	10
$\lvert a, a\rvert^{12}$, $\lvert b, b\rvert^8$, $\lvert d, d\rvert^4$	0	12

In the case of two septimics, the numbers in our symbolic determinants are 0, 1, 2, 3, 4, 5, 6, 7, from which we can form 70 determinant identities of Type 1. They are distributed as follows:

Table 2.

Wt.	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
No.	1	1	2	3	5	5	7	7	8	7	7	5	5	3	2	1	1

Since we require one identity for each coefficient of the same weight as the identity, and our lowest identity is of weight 4, we can have no syzygies con-

taining coefficients of weight less than 4. We have one set of transvectants the weight of whose leading coefficients is 4. The forms in it are of order 16; hence the syzygy will be of order 16. As each term has a weight one greater than the preceding, the syzygy will require one identity of each weight from 4 to 20, inclusive. After deducting these from our total, 53 identities are left, the lowest one having a weight of 6. There can be no syzygy of weight 5. One set of transvectants, of order 12, contains a coefficient of weight 6. To admit one syzygy of this weight will require one identity of each weight from 6 to 18. Our list still contains enough identities for this one syzygy, but it exhausts our identities of weight 6 and 18; hence we can admit but one such syzygy. After it is formed, there are left 40 identities, in which there is *one* only of the lowest weight 7. We can therefore admit one syzygy of weight 7; it is of order 10, so that it uses up one identity of each weight from 7 to 17, leaving 29 identities, the lowest weight being 8, of which there are *two* identities. Since we have two identities of weight 8, we may have two syzygies of that weight. These are of order 8, and so will use up two identities of each weight from 8 to 16; and we have 11 identities remaining. Again there are two identities of the lowest weight 10, so that we may have two syzygies of weight 10. They are of order 4, and so use up ten of the identities, leaving one identity of weight 12, which is all we need to admit one syzygy of order zero and weight 12.

The distribution of the identities in the various terms of the syzygies, as well as the number of syzygies, shows up to advantage in Table 3. The numbers at the heads of the columns indicate the weights of the identities numbered. The footings of the columns must of necessity be identical with the number of identities of each weight given in Table 2.

Table 3.

Syz.	Ord.	Wt.	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I.	16	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
II.	12	6			1	1	1	1	1	1	1	1	1	1	1	1	1		
III.	10	7				1	1	1	1	1	1	1	1	1	1	1			
IV–1.	8	8					1	1	1	1	1	1	1	1	1				
IV–2.	8	8					1	1	1	1	1	1	1	1	1				
V–1.	4	10							1	1	1	1	1						
V–2.	4	10							1	1	1	1	1						
VI.	0	12									1								
Totals			1	1	2	3	5	5	7	7	8	7	7	5	5	3	2	1	1

XVI. *Enumeration of the Quadratic Syzygies for Three Binary Sextics.*

The linear combinants are:

$$(At)^{12},\ (Bt)^{8},\ (Ct)^{6},\ (Dt)^{4},\ (Ft)^{0},$$

of weight 0, 2, 3, 4, 6,

The enumeration of the transvectants is given in Table 1, which follows.

Table 1.

Transvectants.

No.	Ord.	Wt.
1	24	0
2	20	2
2	18	3
5	16	4
4	14	5
9	12	6
6	10	7
11	8	8
6	6	9
9	4	10
2	2	11
5	0	12
62	Total.	

Table 2.

Identities.

No.	Wt.
1	4
2	5
4	6
6	7
9	8
11	9
14	10
15	11
16	12
15	13
14	14
11	15
9	16
6	17
4	18
2	19
1	20
140	Total.

Table 3.

Syz.	Ord.	Wt.	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I.	16	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
II.	14	5		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
III–1.	12	6			1	1	1	1	1	1	1	1	1	1	1	1	1		
III–2.					1	1	1	1	1	1	1	1	1	1	1	1	1		
IV–1.	10	7				1	1	1	1	1	1	1	1	1	1	1			
IV–2.						1	1	1	1	1	1	1	1	1	1	1			
V–1.	8	8					1	1	1	1	1	1	1	1	1				
V–2.							1	1	1	1	1	1	1	1	1				
V–3.							1	1	1	1	1	1	1	1	1				
VI–1.	6	9						1	1	1	1	1	1	1					
VI–2.								1	1	1	1	1	1	1					
VII–1.	4	10							1	1	1	1	1						
VII–2.									1	1	1	1	1						
VII–3.									1	1	1	1	1						
VIII.	2	11								1	1	1							
IX.	0	12									1								
Totals			1	2	4	6	9	11	14	15	16	15	14	11	9	6	4	2	1

In this case we have two types of determinant identities. There are 105 of Type 2, each counting for one condition, and seven of Type 3, each representing five independent identities. Hence, we have altogether 140 independent quadratic identities whose weights are distributed as in Table 2.

Without repeating such discussion as we gave in the previous section, we append herewith a Table 3, similar to that of Section XV, which shows the number of syzygies of each weight and order that can occur.

XVII. *Enumeration of the Quadratic Syzygies for Three Binary Septimics.*

The linear combinants are:

$$(At)^{15},\ (Bt)^{11},\ (Ct)^{9},\ (Dt)^{7},\ (Et)^{5},\ (Ft)^{3},$$

of weight $\quad 0, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6.$

There are in all 128 transvectant forms from which to form the syzygies. They are enumerated in Table 1.

The 420 independent identities are noted in Table 2. There are 280 identities of Type 2, and 28 forms of Type 3, which are equivalent to 140 independent identities.

A Table 3, similar to that presented in the two sections immediately preceding this, is given here for the tabular enumeration of the syzygies.

Table 1.

Transvectants.

No.	Ord.	Wt.
1	30	0
2	26	2
2	24	3
5	22	4
5	20	5
10	18	6
9	16	7
15	14	8
13	12	9
18	10	10
13	8	11
17	6	12
8	4	13
10	2	14
128	Total.	

Table 2.

Identities.

No.	Wt.	No.	Wt.
1	4	1	26
2	5	2	25
4	6	4	24
7	7	7	23
11	8	11	22
15	9	15	21
21	10	21	20
26	11	26	19
31	12	31	18
35	13	35	17
38	14	38	16
38	15		

Total identities, 420.

Table 3.

Syz.	Ord.	Wt.	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
I.	22	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
II.	20	5		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
III–1.	18	6			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
III–2.					1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
IV–1.	16	7				1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1			
IV–2.						1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1			
IV–3.						1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1			
V–1.	14	8					1	1	1	1	1	1	1	1	1	1	1	1	1	1	1				
V–2.							1	1	1	1	1	1	1	1	1	1	1	1	1	1	1				
V–3.							1	1	1	1	1	1	1	1	1	1	1	1	1	1	1				
V–4.							1	1	1	1	1	1	1	1	1	1	1	1	1	1	1				
VI–1.	12	9						1	1	1	1	1	1	1	1	1	1	1	1	1					
VI–2.								1	1	1	1	1	1	1	1	1	1	1	1	1					
VI–3.								1	1	1	1	1	1	1	1	1	1	1	1	1					
VI–4.								1	1	1	1	1	1	1	1	1	1	1	1	1					
VII–1.	10	10							1	1	1	1	1	1	1	1	1	1	1						
VII–2.									1	1	1	1	1	1	1	1	1	1	1						
VII–3.									1	1	1	1	1	1	1	1	1	1	1						
VII–4.									1	1	1	1	1	1	1	1	1	1	1						
VII–5.									1	1	1	1	1	1	1	1	1	1	1						
VII–6.									1	1	1	1	1	1	1	1	1	1	1						
VIII–1.	8	11								1	1	1	1	1	1	1	1	1							
VIII–2.										1	1	1	1	1	1	1	1	1							
VIII–3.										1	1	1	1	1	1	1	1	1							
VIII–4.										1	1	1	1	1	1	1	1	1							
VIII–5.										1	1	1	1	1	1	1	1	1							
IX–1.	6	12									1	1	1	1	1	1	1								
IX–2.											1	1	1	1	1	1	1								
IX–3.											1	1	1	1	1	1	1								
IX–4.											1	1	1	1	1	1	1								
IX–5.											1	1	1	1	1	1	1								
X–1.	4	13										1	1	1	1	1									
X–2.												1	1	1	1	1									
X–3.												1	1	1	1	1									
X–4.												1	1	1	1	1									
XI–1.	2	14											1	1	1										
XI–2.													1	1	1										
XI–3.													1	1	1										
Totals			1	2	4	7	11	15	21	26	31	35	38	38	38	35	31	26	21	15	11	7	4	2	1

XVIII. *Bibliography.*

We have appended here a list of the mathematical papers that have to do with the combinant theory. Much of great value aside from these papers has been found in the various standard texts on invariants and covariants, among which may be mentioned particularly: Grace and Young, "Algebra of Invariants"; Clebsch, "Binäre Formen"; Study, "Ternaere Formen"; and Gordan, "Invariantentheorie."

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